A fundamental differential system of Riemannian geometry

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Abstract

We study a fundamental exterior differential system associated to any given oriented Riemannian manifold M of any dimension. The system was first considered in hypersurface theory of flat Euclidean space, but here it is defined invariantly on the tangent sphere bundle of the given Riemannian manifold. We deduce the structure equations and their main properties. In particular we write a new equivalent equation for the condition of M being an Einstein manifold.

Key Words: tangent sphere bundle, exterior differential system, Euler-Lagrange systems

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1 Introduction

It is a remarkable feature of differential geometry that so many problems have been addressed through the exterior algebra of differential forms and partial differential equations, the theory of exterior differential systems. In the present article we are particularly interested in contact geometry, a branch which interacts strongly with Riemannian geometry. We start by recalling the well-known contact differential system, generated by a natural non-vanishing 1-form θ, on the radius $s > 0$ tangent sphere bundle $SM_s \longrightarrow M$ of a given smooth oriented Riemannian $n+1$-dimensional manifold M. Then we turn to our main purpose, which is to present the

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study of a set of natural n-forms $\alpha_0,\alpha_1,\ldots,\alpha_n$ existing always on SM_s, apparently known to a few geometers but which the author rediscovered independently.

Each of these n-forms on the contact $2n+1$-manifold (SM_s,θ) and their C^∞ linear combinations assume the natural role of Lagrangian forms. So they induce variational principles of the underlying exterior differential system. We believe the study of the Lagrangians α_i, $0\leq i\leq n$, may be pursued through many fields. To be more explicit in this first survey, suppose we have a (Cartan) coframe of SM_1, $e^0,e^1,\ldots,e^n,e^{n+1},\ldots,e^{2n}$, where $\theta=e^0$, the 1-forms $e^0,e^1,\ldots,e^n$ are horizontal and the remaining are vertical (this is the usual terminology of fibre bundle tangent structure and will be explained in section 1). Then in case $n=1$, we have a global coframing (which was already used by Darboux) of θ plus the two 1-forms $\alpha_0=e^1$ and $\alpha_1=e^2$. In case $n=2$,

$$\alpha_0=e^{12},\qquad \alpha_1=e^{14}+e^{32},\qquad \alpha_2=e^{34}. \tag{1.1}$$

In case $n=3$,

$$\alpha_0=e^{123},\qquad \alpha_1=e^{126}+e^{234}+e^{315},\qquad \alpha_2=e^{156}+e^{264}+e^{345},\qquad \alpha_3=e^{456}. \tag{1.2}$$

In this article we show how these n-forms relate to certain calibrated geometries and, at least, to one special Riemannian geometry. The latter consists of a natural G_2 structure existing always on SM_1 for any oriented 4-manifold M. Discovered in [AS09, AS10], it brings new insight on the role of Einstein metrics.

Here we prove a new Theorem, in a general framework, whose content is as follows: an oriented Riemannian $n+1$-manifold is Einstein if and only if α_{n-2} is coclosed. We trust this may be quite useful in the field of Einstein metrics.

For instance, for any given metric of constant sectional curvature k we have the formula

$$\mathrm{d}\alpha_i=\theta\wedge((n+1)\alpha_{i+1}-k(n-i+1)\alpha_{i-1}). \tag{1.3}$$

This is deduced immediately from Theorem 2.1. Throughout the text the reader will notice that we try to explore some of the consequences of (1.3). We look forward to develop the interplay with submanifold theory of space-forms, in a new article, since this seems to be the a most promising feature of the n-forms.

Further applications of the natural Lagrangians go through an analysis of the Euler-Lagrange equations of the first few, say when $i=0,1,2$, functionals $\mathcal{F}_i(N)=\int_{\hat{N}}\alpha_i$ on the set of submanifolds $N\hookrightarrow M$ with lifts $\hat{N}$ to SM_1 (as applied by known references in metric problems of flat ambient space M or in cases of low dimension). We recall in particular the linear Weingarten equations, which we dare to explore in guidance with the study in [BGG03].

To the best of our knowledge, there exist only a few references about the differential system of θ and the α_i. The treatment is rather distant from that which we propose here (for instance we define the forms globally from the beginning) and, seemingly, the n-forms have only been considered having in view the solution of practical problems. Already in Phillip Griffiths' remarkable work [Gri83] we are presented with the forms in the cases of 2 and 3 dimensional base M. In [BCG$^+$91] the emphasis is on an application to a metric and algebraic geometry problem in dimension three (cf. p. 152), the same being true with later articles seen in

[Gri03]. Coming to more recent consulted works, in [IL03] the n-forms are again perceived in low dimensional problems. So it seems the general case only appears for the first time in [BGG03, p. 32], in the discussion of contact differential systems dealing with hypersurface Riemannian geometry. This extraordinary reference introduces the reader to essentially the same differential forms α_i — it is not difficult to understand they are the same objects we define here, regardless the Euclidean setting and the underlying or supporting fibre bundle. The curious formula of the $\mathrm{d}\alpha_i$ for the case $k = 0$ in (1.3) is thus already known.

From what we have realized through the literature, for the importance we give to the subject and the coincidence of the mathematician appearing in all references which mention explicitly the differential system, the author suggests and uses the name *Griffiths forms* to refer to the n-forms α_i of the natural exterior differential system of a tangent sphere bundle of an $n+1$-manifold. The latter is henceforth called the *Griffiths exterior differential system*.

The contents of this article are as follows. In section 2 we present our techniques with the Riemannian geometry of SM_s. We have in view the description of the Griffiths forms and their first structural equations. In section 3 we present a few applications and examples, namely to Einstein metrics. There we also look into special Riemannian structures and concentrate on some variational problems of the geometry of hypersurfaces, a case in which we mostly follow [BGG03]. In section 4 we call attention upon the study of the infinitesimal symmetries and conservation laws. Finally in section 5 we complete the more computational proofs from previous sections.

2 The natural exterior differential system on SM_s

2.1 Geometry of the tangent sphere bundle

Let M be an $n+1$-dimensional Riemannian manifold with metric tensor $g = \langle\ ,\ \rangle$. We need to recall some differential geometry techniques for the study of the total space of the tangent bundle $\pi : TM \longrightarrow M$. The metric techniques, studied below in a second phase, have been thoroughly used and developed in various works by enumerable mathematicians in the last five decades, of which the most well-known are presumably S. Sasaki, P. Dombrovsky and O. Kowalsky. One may include some previous articles of the author in regard to this long and extensive study. The tangent bundle of a given manifold is indeed a proficuous object in the geometrical context and thence a tool which conveys all analytical fields of application.

The manifold TM is well-known to be a $2n+2$-dimensional oriented manifold. A canonical atlas arising from any given atlas of M induces a natural isomorphism $V = \ker \mathrm{d}\pi \simeq \pi^*TM$. This agrees clearly fibre-wise with the tangent bundle to the fibres of TM. Only supposing a linear connection ∇ is given on M, we may say the tangent bundle of TM splits as $TTM = H \oplus V$, where H is a sub-vector bundle (depending on ∇). Clearly the *horizontal* sub-bundle H is also isomorphic to π^*TM through the map $\mathrm{d}\pi$. We may thus define an endomorphism

$$B : TTM \longrightarrow TTM \tag{2.1}$$

transforming H in V, completing $\mathrm{d}\pi$, and vanishing on the *vertical* sub-bundle V. There also exists a canonical vector field ξ over TM defined by $\xi_u = u,\ \forall u \in TM$. Note ξ is independent of the connection and the vector ξ_u lies in the vertical side V_u. Henceforth there exists a unique

horizontal canonical vector field $B^t\xi \in H$ such that $B(B^t\xi) = \xi$. Such vector field is called the *geodesic spray* of the connection, cf. [Sak96]. In the sequel we let $\nabla^* = \pi^*\nabla$ denote the pull-back connection on π^*TM and let $(\cdot)^h$, $(\cdot)^v$ denote the projections of tangent vectors onto their H and V components. Then, $\forall w \in TTM$,

$$\nabla^*_w\xi = w^v \qquad\qquad \text{and} \qquad\qquad H = \ker(\nabla^*_\cdot\xi). \tag{2.2}$$

The manifold TM also inherits a linear connection, denoted ∇^*, which is just

$$\nabla^* \oplus \nabla^*$$

preserving the canonical splitting. Of course, the connecting endomorphism B is parallel for such ∇^*. The theory tells us that, furthermore, for a torsion-free connection ∇, the torsion of ∇^* is given, $\forall v, w \in TTM$, by

$$T^{\nabla^*}(v,w) = R^{\nabla^*}(v,w)\xi = \pi^*R^{\nabla}(v,w)\xi := \mathcal{R}^\xi(v,w) \tag{2.3}$$

where R denotes a curvature tensor (the proof is recalled in section 5). This means the torsion is vertically valued and only depends on the horizontal vectors. The second identity follows by tensoriality and in the third we have defined the tensor $\mathcal{R}^\xi \in \Omega^2_{TM}(V)$.

Now we start using the given metric g of M. We recall the Sasaki metric on TM, also denoted by g, which is given naturally by the pull-back of the metric on M both to H and V. The morphism $B_| : H \to V$ is then an isometry. At this point it is admissible the notation B^t for the adjoint endomorphism of B. Moreover, we remark $J = B - B^t$ is well-known to be an almost complex structure on TM, proving our manifold is indeed always oriented. Notice $\nabla g = 0$ implies $\nabla^* g = 0$.

From now on we assume the given linear connection ∇ is the Levi-Civita connection of M (though the theory may be extended to any metric connection with torsion).

Finally we are ready to consider the tangent sphere bundle SM_s with fixed constant radius $s > 0$,

$$SM_s = \big\{u \in TM : \ \|u\| = s\big\}. \tag{2.4}$$

This hypersurface is also given by the equation $\langle\xi,\xi\rangle = s^2$, thence $TSM_s = \xi^\perp \subset TTM$. Since the manifold TM is orientable, SM_s is also always orientable (the restriction of $\xi/\|\xi\|$ being a unit normal). Moreover, for any $u \in TM\backslash 0$, we may find a local horizontal orthonormal frame $e_0, \ldots, e_n$, on a neighbourhood of $u \neq 0$, such that $e_0 = B^t\xi/\|\xi\|$ (this relies on the smoothness of the Gram-Schmidt process). Note that any frame in H extended with its *mirror* in V clearly determines an orientation on the manifold TM. We adopt the order 'H then V', which makes a difference with its reverse when $\dim M$ is odd.

We are always going to assume M is oriented (though for most purposes the mere existence of a parallel $n+1$-form is sufficient). We let α denote the n-form on TM which is defined as the interior product of $\xi/\|\xi\|$ with the vertical pull-back of the volume form of M. We let vol denote the (usual) pull-back by π of the volume form of M. With the dual coframing $\{e^0, e^1, \ldots, e^n\}$, where $e^0 = e_0^\flat$, clearly the identity $\mathrm{vol} = e^0 \wedge \cdots \wedge e^n$ is verified. Adding the mirror subset $\{\frac{\xi^\flat}{\|\xi\|}, e^{n+1}, \ldots, e^{2n}\}$, with $e^{n+i} = e^i \circ B^t$, $\forall i \geq 1$, we may fix the volume form of TM:

$$\mathrm{Vol} = e^0 \wedge e^1 \wedge \cdots \wedge e^n \wedge \frac{\xi^\flat}{\|\xi\|} \wedge e^{n+1} \wedge \cdots \wedge e^{(2n)} = \frac{(-1)^{n+1}}{s}\xi^\flat \wedge \mathrm{vol} \wedge \alpha. \tag{2.5}$$

Hence, having chosen $+$ or $-\xi/s$ as unit normal direction (according with n odd or even), the canonical orientation $(\pm\xi/s)\lrcorner\mathrm{Vol}$ of the Riemannian submanifold SM_s agrees with $\mathrm{vol}\wedge\alpha = e^{01\cdots(2n)} = e^0\wedge e^1\wedge\cdots\wedge e^{(2n)}$. We shall assume always the canonical orientation $\mathrm{vol}\wedge\alpha$ on SM_s (the same symbols denote the restriction of those forms to the submanifold in cause). A direct orthonormal frame as the one introduced here will be said to be *adapted*.

To facilitate notation we let $SM = SM_s$ with any freely chosen s, only recalled when necessary.

The manifold SM admits a metric linear connection $\nabla^\star$, as we shall see next. For any vector fields y, z on SM, the covariant derivative $\nabla^*_y z$ is well-defined and, admitting y, z perpendicular to ξ, we just have to add a correction term:

$$\nabla^\star_y z = \nabla^*_y z - \frac{1}{s^2}\langle\nabla^*_y z, \xi\rangle\xi = \nabla^*_y z + \frac{1}{s^2}\langle y^v, z^v\rangle\xi. \tag{2.6}$$

Since $\langle\mathcal{R}^\xi(y,z),\xi\rangle = 0$, we see from (2.3) that a torsion-free connection D is given by $D_y z = \nabla^\star_y z - \frac{1}{2}\mathcal{R}^\xi(y,z)$. The reader should be aware D is not the Levi-Civita connection if $R^\nabla \neq 0$; one may further consult [Alb10, Alb11] for details on metric connections on SM.

2.2 The contact structure and new n-forms

Continuing to explore the ideas and notation introduced above, we let θ denote the 1-form on SM

$$\theta = s\,e^0 = (B^t\xi)^\flat = \langle\xi, B(\,\cdot\,)\rangle. \tag{2.7}$$

We wish to make further claims on the natural geometry of SM. They are based on the following Proposition, whose proof shall be recalled in section 5. The result was essentially deduced by Y. Tashiro in the late 1960's through chart computations, cf. [Bla02].

Proposition 2.1. *We have* $\mathrm{d}\theta = e^{(1+n)1} + \cdots + e^{(2n)n}$. *Equivalently,* $\forall v, w \in TSM$,

$$\mathrm{d}\theta(v,w) = \langle v, Bw\rangle - \langle w, Bv\rangle. \tag{2.8}$$

It is easy to see that (SM,θ) is a contact manifold. The same is to prove that $\theta\wedge(\mathrm{d}\theta)^n = (-1)^{\frac{n(n+1)}{2}} n! s\,\mathrm{vol}\wedge\alpha \neq 0$, as we shall care to establish later. We ask the reader to accept our abbreviation of the wedge product, which happens only when there seems no danger of being misled. We also observe here that the expression of $\mathrm{d}\theta$ is not linear in s, lest the reader should be driven to conclude otherwise.

Remark. We may also describe a *metric* contact structure on SM. Finding the correct weights on the fixed metric, the 1-form θ and the so-called Reeb vector field, which is of course a multiple of $B^t\xi$, gives

$$\hat{g} = \frac{1}{4s^2}g, \qquad \hat{\xi} = 2B^t\xi, \qquad \eta = \hat{g}(\hat{\xi},\,\cdot\,) = \frac{1}{2s^2}\theta, \qquad \varphi = B - B^t - 2\xi\otimes\eta. \tag{2.9}$$

And then we have $\eta(\hat{\xi}) = 1$, $\varphi(\hat{\xi}) = 0$, $\varphi^2 = -1 + \eta\otimes\hat{\xi}$, $\hat{g}(\varphi\,\cdot\,,\varphi\,\cdot\,) = \hat{g} - \eta\otimes\eta$ and $\mathrm{d}\eta = 2\hat{g}(\,\cdot\,,\varphi\,\cdot\,)$ as we wished. This metric contact structure is Sasakian if and only if M has constant sectional curvature $1/s^2$.

Now we are able to define $n+1$ natural n-forms on SM with any constant radius s. First, for $0\leq i\leq n$, we let the rational n_i be given by

$$n_i=\frac{1}{i!(n-i)!}. \tag{2.10}$$

Continuing with the notation and the adapted frame introduced earlier, we then define α_n:

$$\alpha_n=\alpha=\frac{\xi}{\|\xi\|}\lrcorner(\pi^{-1}\mathrm{vol}_M)=e^{(n+1)}\cdots e^{(2n)} \tag{2.11}$$

where $\pi^{-1}\mathrm{vol}_M$ is the vertical pull-back of the volume form of M (overall distinguished from $\pi^*\mathrm{vol}_M=\mathrm{vol}$). Finally for each i we define n-forms α_i by

$$\alpha_i=n_i\,\alpha\circ(B^{n-i}\wedge 1^i_{TSM}), \tag{2.12}$$

this is, $\forall v_1,\ldots,v_n\in TSM$,

$$\alpha_i(v_1,\ldots,v_n)=n_i\sum_{\sigma\in S_n}\mathrm{sg}(\sigma)\alpha(Bv_{\sigma_1},\ldots,Bv_{\sigma_{n-i}},v_{\sigma_{n-i+1}},\ldots,v_{\sigma_n}). \tag{2.13}$$

Note that $B^{n-i}=\wedge^{n-i}B=B\wedge\cdots\wedge B$ with $n-i$ factors. The map $1=1_{TSM}$ denotes the identity endomorphism of TSM. The notation $\alpha\circ(B^{n-i}\wedge 1^i)$ shall be duly introduced and justified in section 5. Notice α_n is unambiguously defined, because $\alpha\circ\wedge^n 1=n!\alpha$ and $n_n=n!^{-1}$. We remark also that $\alpha_0=e^{1\ldots n}$ (which justifies the definition of the weight n_i). For convenience we define $\alpha_{n+1}=0$.

The reader may see all the α_i for the cases $n=1,2,3$ in the introduction section.

Proposition 2.2 (Basic structure equations)**.** *For any $0\leq i\leq n$ we have:*

$$*\,\theta=s\,\alpha_0\wedge\alpha=\frac{s(-1)^{\frac{n(n+1)}{2}}}{n!}\,(\mathrm{d}\theta)^n, \tag{2.14}$$

$$*\,(\mathrm{d}\theta)^i=(-1)^{\frac{n(n+1)}{2}}\frac{i!}{(n-i)!s}\,\theta\wedge(\mathrm{d}\theta)^{n-i},\qquad *\alpha_i=\frac{(-1)^{n-i}}{s}\,\theta\wedge\alpha_{n-i}. \tag{2.15}$$

Moreover, $\alpha_i\wedge\mathrm{d}\theta=0$ and $\alpha_i\wedge\alpha_j=0,\ \forall j\neq n-i$.

Of course $*$ denotes the Hodge star-operator on SM. Recall $**=1_{\Lambda^*_{SM}}$.

Theorem 2.1 (1st-order structure equations)**.** *We have*

$$\mathrm{d}\alpha_i=\frac{1}{s^2}(i+1)\theta\wedge\alpha_{i+1}+\mathcal{R}^\xi\alpha_i \tag{2.16}$$

where

$$\mathcal{R}^\xi\alpha_i=\sum_{0\leq j<q\leq n}\sum_{p=1}^{n}sR_{jq0p}\,e^{jq}\wedge e_{p+n}\lrcorner\alpha_i \tag{2.17}$$

The proof of this and the following result is postponed to section 5. Notice in particular the formulas $\mathcal{R}^\xi\alpha_0=0,\ \mathcal{R}^\xi\alpha_1=-sr\,\mathrm{vol}$. In other words,

$$\mathrm{d}\alpha_0=\frac{1}{s^2}\theta\wedge\alpha_1, \tag{2.18}$$

$$\mathrm{d}\alpha_1=\frac{2}{s^2}\theta\wedge\alpha_2-\frac{r}{s}\,\mathrm{vol} \tag{2.19}$$

where $r=\mathrm{Ric}\,(\xi,\xi)$ is a C^∞ function on SM determined by the Ricci curvature of M. More precisely, r is defined by

$$u\in SM\ \longmapsto\ \mathrm{Ric}_{\,\pi(u)}(u,u)=\mathrm{Tr}\,R_{\pi(u)}(\,\cdot\,,u)u=s^2\sum_{j=1}^n R_{j00j}. \tag{2.20}$$

We remark

$$\mathrm{d}\alpha_n=\mathcal{R}^\xi\alpha_n=\sum_{0\leq j<q\leq n}\sum_{p=1}^n(-1)^{p-1}sR_{jq0p}\,e^{jq}\wedge e^{(n+1)\ldots\widehat{(n+p)}\ldots(2n)}. \tag{2.21}$$

Note that

$$\mathrm{d}(\mathcal{R}^\xi\alpha_i)=\frac{1}{s^2}(i+1)\theta\wedge\mathcal{R}^\xi\alpha_{i+1} \tag{2.22}$$

and so

$$\mathrm{d}\theta\wedge\mathcal{R}^\xi\alpha_i=0. \tag{2.23}$$

It is trivial to see

$$\delta\theta=0. \tag{2.24}$$

Proposition 2.3. *The differential forms α_n and α_{n-1} are always coclosed.*

Proof. $\mathrm{d}{*}\alpha_{n-1}=\frac{-1}{s}\mathrm{d}(\theta\wedge\alpha_1)=\frac{-1}{s}\mathrm{d}\theta\wedge\alpha_1+\frac{1}{s}\theta\wedge\mathrm{d}\alpha_1=0$ and $\mathrm{d}{*}\alpha_n=\mathrm{d}\,\mathrm{vol}=\pi^*\mathrm{d}\,\mathrm{vol}_M=0$. ∎

We call the set of differential forms $\theta,\alpha_1,\ldots,\alpha_n$ and the d-closed ideal it generates the *natural exterior differential system of M on its tangent sphere bundle* or the *Griffiths exterior differential system* of M. The n-forms α_i are called *Griffiths forms.*

3 Applications, examples and Euler-Lagrange equations

3.1 Examples and open problems

Let us see some examples from which we can read the fundamental equations of the $SO(n)$-structure of a tangent sphere bundle $SM\rightarrow M$ of a given oriented Riemannian manifold M of dimension $n+1$. Let us thus see the examples of the proposed Griffiths system of M.

EXAMPLE 1.I. Suppose $n=1$ and $s=1$. We then have a global coframing on SM_1 given by

$$\theta=e^0,\qquad \alpha_0=e^1,\qquad \alpha_1=\alpha=e^2,$$

This reflects the triviality of the principal S^1-bundle and the natural exterior differential system agrees with Cartan structure equations:

$$\mathrm{d}\theta = \alpha_1 \wedge \alpha_0, \qquad \mathrm{d}\alpha_1 = r\,\alpha_0 \wedge \theta, \qquad \mathrm{d}\alpha_0 = \theta \wedge \alpha_1 \tag{3.1}$$

where $r = R_{1001}$ is the Gauss curvature of M.

Example 1.ii. One becomes curious in seeing the above equations in the trivial case of $S\mathbb{R}^2_1 = \mathbb{R}^2 \times S^1$. Admitting coordinate functions (x^1, x^2, u_1, u_2), subject to $(u_1)^2 + (u_2)^2 = 1$, we immediately find $e^0 = u_1\mathrm{d}x^1 + u_2\mathrm{d}x^2$, $e^1 = -u_2\mathrm{d}x^1 + u_1\mathrm{d}x^2$, $e^2 = -u_2\mathrm{d}u_1 + u_1\mathrm{d}u_2$. Using the identity $u_1\mathrm{d}u_1 + u_2\mathrm{d}u_2 = 0$, the equations above *do* follow just like for any flat surface.

Example 2. Suppose $n = 2$ and $s = 1$. Then the equations for α_i, $i = 0, 1, 2$, give us the global tensors $\mathcal{R}^\xi\alpha_0 = 0$, $\mathcal{R}^\xi\alpha_1 = -r\,\mathrm{vol}$ (cf. (2.20)) and

$$\mathcal{R}^\xi\alpha_2 = R_{0101}e^{014} + R_{0201}e^{024} + R_{1201}e^{124} - R_{0102}e^{013} - R_{0202}e^{023} - R_{1202}e^{123}. \tag{3.2}$$

These illustrative expressions shall be discussed later. In [Gri03, p. 461] we find another approach to the equations, via principal frame bundle of M, having in view an example of a hyperbolic exterior differential system.

Example 3. It is quite interesting to consider the case of constant sectional curvature k in any dimension $n+1$. Since the Riemann curvature tensor is $R_{ijpq} = k(\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq})$, we prove in section 5 that $\mathcal{R}^\xi\alpha_i = -k(n-i+1)\theta \wedge \alpha_{i-1}$. In other words,

$$\mathrm{d}\alpha_i = \theta \wedge \Big(\frac{1}{s^2}(i+1)\alpha_{i+1} - k(n-i+1)\alpha_{i-1}\Big). \tag{3.3}$$

Like α_{n+1}, we define $\alpha_{-1} = 0$. Notice $\mathcal{R}^\xi\alpha_1 = -snk\,\mathrm{vol}$, just as expected through (2.19). Regardless the awkward context, we may formally compare the deduced formula with the *Frenet equations* of a curve in $\mathbb{R}^n$ described in [Gri03, p. 23]. Furthermore it is easy to see that, for all i,

$$\mathrm{d} * \alpha_i = 0 \tag{3.4}$$

in the case of constant sectional curvature metric.

Now let us see an application which was found well before the present construction of the natural exterior differential system of a Riemannian manifold. We shall need to refer to concepts of G_2 geometry, which the reader may follow e.g. in [Joy09].

In [AS09, AS10] it was proved that the total space of the unit tangent sphere bundle $SM_1 \to M$ of any given oriented Riemannian 4-manifold M carries a natural G_2-structure. The structure is discovered through a geometrical and algebraic reasoning, quite easy to describe as the reader may care to verify. It led us conveniently to introduce the 3-forms $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, precisely by formula (2.12), and to deduce the respective structural differential equations (also cf. [Alb10]). The space is now called *G_2-twistor* or *gwistor space* and its fundamental G_2-structure form is shown to be

$$\phi = \theta \wedge \mathrm{d}\theta + \alpha_1 - \alpha_3. \tag{3.5}$$

Our original incursion in Riemannian exceptional geometry was rather fortunate with the Hodge dual of ϕ. The structure form ϕ is coclosed (a well-known condition which takes the

G_2 structure to be called cocalibrated) if and only if the given Riemannian manifold M is Einstein. We also aim strongly today for a complete study of the tensors $\mathcal{R}^\xi\alpha_2, \mathcal{R}^\xi\alpha_3$.

It is appropriate to recall the canonical decomposition under representation theory of the orthogonal group $SO(n)$, due to the new study we have undertaken,

$$\Lambda^n(\mathbb{R}^{2n+1}) = \bigoplus_{p+q+l=n} \Lambda^p(\mathbb{R}^n)\otimes\Lambda^q(\mathbb{R}^n)\otimes\Lambda^l(\mathbb{R}). \tag{3.6}$$

An open problem is to find the conditions for a linear combination $\varphi = \sum_{i=0}^n b_i\alpha_i + c\theta^\varepsilon\wedge(\mathrm{d}\theta)^{[\frac{n}{2}]}$, with $b_i, c \in \mathrm{C}^\infty_{SM_1}$ and $\varepsilon = 0$ or 1 according to the parity of n, to be a calibration of degree n. We recall a calibration is a closed p-form φ such that $\varphi_{|V} \leq \mathrm{vol}_V$ for every oriented tangent p-plane V.

For n even we certainly have an obvious φ of degree n. For $n = 1$ the question may be solved easily recurring to (3.1). For $n = 2$ and 3 we have a complete classification in [Joy09, Theorems 4.3.2 and 4.3.4] of all the possible calibrations which may occur pointwise, in their algebraic form, as elements of $\Lambda^n(\mathbb{R}^{2n+1})$. Then we notice that several cases can be written as the φ we refer above. Others may not — it remains an open problem presumably too long to be ended here.

In case $n = 3$, besides $\theta\wedge\mathrm{d}\theta$ or the G_2 structure of gwistor space shown above, it is interesting to discuss the possibility of a calibration of special-Lagrangian 3-folds precisely as in the referred Theorem in [Joy09]. It appears as the real part of

$$(e^1+\sqrt{-1}e^4)\wedge(e^2+\sqrt{-1}e^5)\wedge(e^3+\sqrt{-1}e^6) = \alpha_0-\alpha_2+\sqrt{-1}(\alpha_1-\alpha_3).$$

The imaginary part is of course also relevant, as well as their Hodge duals. Then we may prove the following result much in the same way of [Alb10].

Theorem 3.1. *Suppose M has dimension four.*
(i) The 3-form $\alpha_0-\alpha_2$ is never closed. It is coclosed if and only if M has constant sectional curvature.
(ii) The 3-form $\alpha_1-\alpha_3$ is never closed. It is coclosed if and only if M is an Einstein manifold.

Sketch of the proof. We apply Proposition 2.2 and Theorem 2.1 directly (cf. Proposition 2.3). The rigidity identities appear immediately. They either allow or imply a curvature tensor such that formula (3.3) is a solution. More concretely, we have $\mathrm{d}*(\alpha_1-\alpha_3) = -\theta\wedge\mathcal{R}^\xi\alpha_2$ and a closer inspection reveals that Einstein equation is the required vanishing condition (also cf. section 3.2). ∎

3.2 Degree reduction of Einstein equation

Now we shall see an equation involving the Ricci tensor of M, again in any dimension $n+1$. Let ρ be the 1-form defined by $\xi\lrcorner\pi^{-1}\mathrm{Ric}$. Note that we refer to the vertical lift of the Ricci 2-tensor, restricted to TSM. In other words, using an adapted frame on SM,

$$\rho = \sum_{a,b=1}^n sR_{ab0a}\,e^{b+n}. \tag{3.7}$$

Note $1 \leq a, b \leq n$ and indeed this is a global 1-form. The next Theorem transforms the equation for an Einstein metric into a formally degree 1 equation. We return to the radius s tangent sphere bundle (for the whole article this important detail was cared only for the completion of the exposition).

Theorem 3.2. *We have*

$$\mathrm{d} * \alpha_{n-2} \;=\; \rho \wedge \mathrm{vol}. \tag{3.8}$$

Hence the metric on M is Einstein if and only if $\delta\alpha_{n-2} = 0$.

Proof. We leave the deduction of the formula to the computations section, section 5 below. The conclusion about the metric being Einstein follows by noticing how ρ is defined. ■

3.3 Recalling Euler-Lagrange systems theory

We wish to relate the study of Euler-Lagrange systems with $(SM, \theta, \mathrm{d}\theta, \alpha_l)$ for some fixed α_l, the Lagrangian form, in the context of Riemannian geometry. The involved theory of exterior differential systems of contact manifolds is beautifully surveyed in [BGG03] as well as in some sections of [Gri03].

We need to recall that theory, so in this section we assume (S, θ) is any given contact manifold, not necessarily metric, of dimension $2n + 1$.

The *contact differential ideal* $\mathcal{I}$ is defined as the d-closed ideal generated by $\theta \in \Omega^*_S$. In other words, it is the ideal algebraically generated by exterior multiples of θ and $\mathrm{d}\theta$. A *Legendre submanifold* of S consists of an n-dimensional manifold N together with an immersion $f : N \to S$ such that $f^*\theta = 0$. The same is to say N is a maximal integral submanifold, — the expression *integral* meaning $f^*\theta = 0$ or $f^*\mathcal{I} = 0$. We also recall that there exists a generalisation of the famous Darboux Theorem of symplectic geometry, which guarantees that certain generic Legendre submanifolds appear as zero sections of $n + 1$ of a set of so-called Pfaff coordinates. These are then called the *transverse* Legendre submanifolds. A Legendre submanifold is C^1-differentiable close to such a generic Legendre submanifold N if it appears as the graph of a function on N in the remaining Pfaff coordinates. The paradigm example (the local model) is the 1-jet manifold of the Euclidean flat space with coordinates z, x^i, p_i, contact form $\theta = \mathrm{d}z - \sum_{i=1}^n p_i \mathrm{d}x^i$ and N given by $z = 0, p_i = 0$.

A *Lagrangian* is simply a $\Lambda \in \Omega^n_S$. It gives rise to a functional on the set of smooth, compact Legendre submanifolds $N \subset S$, possibly with boundary, defined by:

$$\mathcal{F}_\Lambda(N) = \int_N \Lambda \tag{3.9}$$

(the integral is for the restriction or pull-back to N of the Lagrangian, but the relevant point here is that Λ is defined independently.) There are two notions of equivalence for such specified n-forms. An equivalence class $[\Lambda]$ is represented by any element of $\Lambda + \mathcal{I}^n + \mathrm{d}\Omega^{n-1}$, where $\mathcal{I}^n = \mathcal{I} \cap \Omega^n$. Such Lagrangian class clearly induces the same functional on Legendre submanifolds without boundary. On the other hand, an algebraic identity carries over to the whole contact manifold:

$$\mathcal{I}^k = \Omega^k, \qquad \forall\, k > n. \tag{3.10}$$

Hence we have that $\mathrm{d}\Lambda\in\mathcal{I}^{n+1}$ and so the above class is well defined in the cohomology ring of degree n for the differential complex $(\Omega^n/\mathcal{I}^n,\mathrm{d})$. We may rephrase, concluding that the set of representatives $[\Lambda]$ of Lagrangian n-forms corresponds with the *characteristic cohomology* ring $\bar{H}^n(S)$ of the exterior differential system (S,θ). It relates to de Rham cohomology via the short exact sequence $0\to\mathcal{I}^k\to\Omega^k\to\Omega^k/\mathcal{I}^k\to 0$. Having chosen a Lagrangian Λ we wish to study the functional $\mathcal{F}_\Lambda$. By (3.10) there exist two forms α,β on S such that

$$\mathrm{d}\Lambda=\theta\wedge\alpha+\mathrm{d}\theta\wedge\beta=\theta\wedge(\alpha+\mathrm{d}\beta)+\mathrm{d}(\theta\wedge\beta),$$

By [BGG03, Theorem 1.1], there exists a unique global exact form Π such that $\Pi\wedge\theta=0$ and $\Pi\equiv\mathrm{d}\Lambda$ in $\bar{H}^{n+1}(\mathcal{I})$:

$$\Pi=\mathrm{d}(\Lambda-\theta\wedge\beta)=\theta\wedge(\alpha+\mathrm{d}\beta). \tag{3.11}$$

The $n+1$-form Π is called the *Poincaré-Cartan* form.

Now suppose we have a variation of Legendre submanifolds with fixed boundary, i.e. suppose there is a smooth map $F:N\times[0,1]\to S$ such that each $F_t=F_{|N_t}$, with $N_t=N\times\{t\}$, defines a Legendre submanifold (in other words $F^*\theta\equiv 0\mod \mathrm{d}t$) and $\partial(F(N_t))$ is independent of t. The variation is of course in turn of the Legendre submanifold $F_0=f:N\to S$. Then, by subtracting $\int_{N_t}\theta\wedge\beta=0$ to the left hand side, by applying the well-known formula of the usual derivative becoming the Lie derivative under the integral and using the Cartan formula, it is proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{N_t}\Lambda=\int_{N_t}\frac{\partial}{\partial t}\lrcorner\Pi. \tag{3.12}$$

Passing to variational calculus notation, for variational direction vector field $v\in\Gamma_0(N;f^*TS)$ vanishing along ∂N, at point $t=0$ playing the role of $\frac{\partial}{\partial t}$, the previous identity reads

$$\begin{aligned}\delta(\mathcal{F}_\Lambda)_N(v)&=\int_N v\lrcorner f^*\Pi\\&=\int_N(v\lrcorner f^*\theta)f^*\Psi.\end{aligned} \tag{3.13}$$

The last equality follows from the existence, as we saw above, of a non-unique n-form Ψ such that $\Pi=\theta\wedge\Psi$. The conclusion is that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathcal{F}_\Lambda(N_t)=0\qquad\text{if and only if}\qquad f^*\Psi=0. \tag{3.14}$$

A Legendre submanifold satisfying (3.14) is called a *stationary* Legendre submanifold. The exterior differential system algebraically generated by $\theta,\mathrm{d}\theta,\Psi$ is called the *Euler-Lagrange system* of (S,θ,Λ); its Poincaré-Cartan form Π is said to be non-degenerate if it has no other degree-1 factors other than multiples of θ.

In sum, the guiding line to determine the critical points of (3.9) is the computation of the Poincaré-Cartan form (3.11), its transformation as the wedge product $\theta\wedge\Psi$ and finally, due to (3.14), the analysis of condition $f^*\Psi=0$.

We recall a very beautiful example regarding the model contact manifold mentioned at the beginning of this section. It is also from [BGG03]. A classical Lagrangian $\Lambda=L(z,x^i,p_i)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$ on the Euclidean 1-jet manifold gives place to the transverse Legendre submanifolds

of the kind $N=\{(z(x),x,\frac{\partial z}{\partial x^i})\}$ where $x=(x^1,\ldots,x^n)$. Then, letting $L_z=\partial L/\partial z,\ L_{p_i}=\partial L/\partial p_i$,

$$\begin{aligned}\mathrm{d}\Lambda &= L_z\,\theta\wedge\mathrm{d}x+\sum L_{p_i}\,\mathrm{d}p_i\wedge\mathrm{d}x\\ &= \theta\wedge L_z\,\mathrm{d}x-\mathrm{d}\theta\wedge\sum(-1)^{i+1}L_{p_i}\,\mathrm{d}x^1\wedge\cdots\widehat{\mathrm{d}x^i}\cdots\wedge\mathrm{d}x^n,\end{aligned}$$

so by the above prescription ($\Pi=\theta\wedge(\alpha+\mathrm{d}\beta)$)

$$\Pi=\theta\wedge(L_z\,\mathrm{d}x+\mathrm{d}(\sum(-1)^iL_{p_i}\,\mathrm{d}x^1\wedge\cdots\widehat{\mathrm{d}x^i}\cdots\wedge\mathrm{d}x^n))=\theta\wedge\Psi.$$

A transverse Legendre submanifold is stationary if and only if its defining functions satisfy the Euler-Lagrange equations:

$$\frac{\partial L}{\partial z}-\sum\frac{\mathrm{d}}{\mathrm{d}x^i}(\frac{\partial L}{\partial p_i})=0. \tag{3.15}$$

3.4 Euler-Lagrange systems on the unit tangent sphere bundle

Again we consider an oriented $n+1$-dimensional Riemannian manifold M together with its unit tangent sphere bundle $SM_1\xrightarrow{\pi}M$, endowed with the canonical Sasaki metric and metric connection (with torsion) ∇^* induced from the Levi-Civita connection on M.

For the rest of this section we assume the notation $f:N\rightarrow M$ to refer to a compact isometric immersed submanifold of M of dimension n. For simplicity we assume N is oriented, but in regard to the problems below this may be overcome by passing to a double cover.

There exists a smooth lift $\hat{f}:N\rightarrow SM_1$ of f. We simply define $\hat{f}(x)=\vec{\nu}_{f(x)}$, the unique unit normal in $T_{f(x)}M$ chosen according to the orientations of N and M. Note that $\hat{f}$ is also defined on ∂N. It is easy to see that, up to the vector bundle isometry $\mathrm{d}\pi_{|H}:H\rightarrow\pi^*TM$ on the horizontal side, we have the decomposition into horizontal plus vertical:

$$\mathrm{d}\hat{f}(w)=\mathrm{d}f(w)+(f^*\nabla)_wf^*\vec{\nu}. \tag{3.16}$$

Indeed, the vertical part at each point $x\in N$ is $\nabla^*_{\mathrm{d}\hat{f}(w)}\xi=(\hat{f}^*\nabla^*)_w\hat{f}^*\xi$, where ξ is the canonical vertical vector field on SM_1, cf. (2.2). Clearly, $\hat{f}^*\xi_x=\vec{\nu}_{f(x)}=f^*\vec{\nu}_x$ and $\hat{f}^*\pi^*=f^*$.

By definition of $\hat{f}$ we have the phenomena that $\hat{f}:N\rightarrow SM_1$ defines a Legendre submanifold of the natural contact structure: $\hat{f}^*\theta=0$. In other words, $\hat{f}(N)$ is an integral submanifold of θ (and $\mathrm{d}\theta$). If we choose an adapted direct orthonormal coframe $e^0,e^1,\ldots,e^{2n}$ locally on SM_1, then we have also a direct orthonormal coframe $e^1,\ldots,e^n$ for N (we use the same letters, knowing the latter cannot be said to be horizontal). Now, from (3.16), for any $1\leq j\leq n$ we have

$$\hat{f}^*e^j=e^j\qquad\text{and}\qquad\hat{f}^*e^{j+n}=-\sum_{k=1}^{n}A^j_ke^k \tag{3.17}$$

with A the second fundamental form of N. We recall, $A=-\nabla\vec{\nu}:TN\rightarrow TN$. By Proposition 2.1 and $\hat{f}^*\mathrm{d}\theta=\mathrm{d}\hat{f}^*\theta=0$ we confirm in particular that A^j_k is a symmetric tensor.

Conversely, a smooth Legendre submanifold is locally the lift $N\rightarrow SM_1$ of an oriented smooth n-submanifold of $N\hookrightarrow M$ if and only if $\alpha_{0|N}>0$. We are going to need and thence assume this open condition throughout.

We now present a classical, well-known result on minimal surfaces. Our proof is essentially the generalisation of the flat Euclidean case shown in [BGG03] (which uses the Grassmannian bundle of n-planes instead and there invokes the *Legendrian phenomena* seen above); but it is interesting to note it works with any ambient manifold M. Moreover, we easily write the formula for the first-variation of the volume, cf. [Xin03].

Theorem 3.3 (Classical Theorem)**.** *Let N be an isometrically immersed hypersurface in M and let H be the mean curvature vector field, i.e. $H=\frac{1}{n}(A_1^1+\cdots+A_n^n)\vec{\nu}$. Then, $\forall v\in\Gamma(N,f^*TM)$,*

$$\delta(\mathrm{vol})_N(v)=-\int_N n\langle v,H\rangle\,\mathrm{vol}_N. \tag{3.18}$$

In particular, N is minimal if and only if its mean curvature vanishes.

Proof. Recall that a minimal hypersurface is one which is critical for the volume functional. By (3.17) we see immediately that the volume of an n-dimensional submanifold N in M is given by $\mathcal{F}_{\alpha_0}(N)=\int_N\alpha_0$. Now, by Theorem 2.1, the Poincaré-Cartan form of α_0 is just $\mathrm{d}\alpha_0=\theta\wedge\alpha_1=\Pi$. Thus, within the open set of those for which $\alpha_{0N}>0$, a Legendre submanifold in SM_1 is stationary for $\mathcal{F}_{\alpha_0}$ if and only if $\hat{f}^*\alpha_1=0$. Since

$$\alpha_1=e^{1+n}\wedge e^2\wedge\cdots\wedge e^n+e^1\wedge e^{2+n}\wedge e^3\wedge\cdots\wedge e^n+\text{ etc},$$

we have $\hat{f}^*\alpha_1=-(A_1^1+\cdots+A_n^n)\,e^1\wedge\cdots\wedge e^n=-n\langle H,\vec{\nu}\rangle\,\mathrm{vol}_N$. Then we apply (3.13), admitting that $v\lrcorner f^*\theta=\langle v,\vec{\nu}\rangle$. ■

Let us now consider the other Griffiths n-forms α_i defined in section 2.2. They give, in their own right, interesting Lagrangian systems on the contact manifold SM_1. Disregarding the context of a manifold M with, in general, non-vanishing curvature, the analogous Lagrangians referred in [BGG03, p. 32], defined on the Grassmannian bundle of n-planes over Euclidean flat space, tell us of the following common feature. Each pull-back $\hat{f}^*\alpha_i$ to a hypersurface N is the sum of all $i\times i$ minor determinants of A, times $(-1)^i$ and times the volume form of N. In particular, writing $K=\det A$,

$$\int_N\alpha_n=(-1)^n\int_N K\,\mathrm{vol}_N. \tag{3.19}$$

This is the so-called Gauss-Kronecker curvature of N when M is flat. Moreover, by (2.21), if M is flat we have $\mathrm{d}\alpha_n=0$ and thence a variationally trivial functional, i.e. constant under continuity for the C^∞-topology of the space of immersed N.

Let $\sigma_i(A)$ denote the elementary symmetric polynomial of degree i in the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A. The next result resumes the assertion above.

Proposition 3.1. *$\forall 0\leq i\leq n$, we have $\hat{f}^*\alpha_i=(-1)^i\sigma_i(A)\,\mathrm{vol}_N$.*

The proof is quite straightforward, so we leave it to section 5.5. As seen previously, or since σ_1 is the trace, the functional

$$\mathcal{F}_1(N)=\int_N\alpha_1=-n\int_N\|H\|\,\mathrm{vol}_N \tag{3.20}$$

corresponds with the integral of the mean curvature on immersed submanifolds $N\subset M$.

Theorem 3.4. *Suppose our Riemannian manifold M has dimension $n+1>2$. An isometric immersed hypersurface $f:N\to M$ is stationary for the mean curvature functional $\mathcal{F}_1$ if and only if*

$$\mathrm{Scal}^N=\mathrm{Scal}^M-r_{\vec{\nu}} \tag{3.21}$$

where $r_{\vec{\nu}}=\mathrm{Ric}\,(\vec{\nu},\vec{\nu})$ is induced from the Ricci tensor of M and Scal *denotes scalar curvatures. In particular, if M is an Einstein manifold, say with $\mathrm{Ric}=cg$ and c constant, then N has constant scalar curvature $\mathrm{Scal}^N=nc$.*

Proof. Let N be any hypersurface, not necessarily stationary. By Gauss equation, the curvatures of N and M satisfy $R^M_{ijji}=R^N_{ijji}-\lambda_i\lambda_j,\ \ \forall 1\leq i<j\leq n$, in an orthonormal basis diagonalizing A. Hence

$$\mathrm{Scal}^M=\sum_{i,j=0}^n R^M_{ijji}=2r_{\vec{\nu}}+\mathrm{Scal}^N-2\sigma_2(A),$$

which is mostly a well-known formula, since $\sigma_2(A)=\sum_{i<j}\lambda_i\lambda_j=\frac{1}{2}(\|H\|^2n^2-\sum_{i=1}^n\lambda_i^2)$. Now, recurring to $SM_1\to M$, by Theorem 2.1 or (2.19) we know $\mathrm{d}\alpha_1=\theta\wedge(2\alpha_2-r\alpha_0)$. Hence, by (3.14), N is stationary if and only if

$$2\sigma_2(A)-\hat{f}^*r=0,$$

where clearly $\hat{f}^*r$ in the previous notation agrees with $r_{\vec{\nu}}$. The result follows. ■

The Theorem with the hypothesis $c=0$ is known in a close context: constant scalar curvature hypersurfaces N of Euclidean space $\mathbb{R}^{n+1}$ are critical for $\mathcal{F}_1$ when varying within the volume preserving class. Our variational principle, with eventually ∂N fixed, is of greater generality.

Now for Einstein ambient space M, we see through a formula in the last proof that $\mathcal{F}_2$ gives a variational function essentially on the scalar curvature of N.

Theorem 3.5. *Let M be a Riemannian manifold of dimension $n+1>2$ and constant sectional curvature k. Then a compact hypersurface N is a critical point of the scalar curvature functional $\int_N\mathrm{Scal}^N\,\mathrm{vol}_N$ if and only if the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A satisfy (assume $\lambda_3=0$ for $n=2$)*

$$6\sum_{j_1<j_2<j_3}\lambda_{j_1}\lambda_{j_2}\lambda_{j_3}+k(n-1)(n-2)(\lambda_1+\cdots+\lambda_n)=0. \tag{3.22}$$

In other words, $6\sigma_3(A)+kn(n-1)(n-2)\|H\|=0$.

Proof. As seen above, we have $\mathrm{Scal}^N=\mathrm{Scal}^M-2r+2\sigma_2(A)$ corresponding with the Lagrangian $\Lambda=((n+1)nk-2nk)\alpha_0+2\alpha_2$. Recurring to (3.3) we find

$$\begin{aligned}\mathrm{d}\Lambda &= (n-1)nk\theta\wedge\alpha_1+2\theta\wedge(3\alpha_3-k(n-1)\alpha_1)\\ &= \theta\wedge(6\alpha_3+k(n-1)(n-2)\alpha_1)\end{aligned}$$

and the result now follows easily. ■

Note the case $n = 2$ is always satisfied and invariant of the ambient manifold as expected by Gauss-Bonnet Theorem.

We may finally define the functionals on the set of compact immersed hypersurfaces of M:

$$\mathcal{F}_i(N) = \int_N \alpha_i. \tag{3.23}$$

Notice it is the same functional for Legendre hypersurfaces with $\alpha_{0|N} > 0$ on SM_1. As seen above, $\mathcal{F}_0, \mathcal{F}_1$ are always averaging the volume and the mean curvature of N, respectively, up to a constant factor. For $n = 1$, we remark $\mathcal{F}_0$ gives unparametrised geodesics as length stationary submanifolds and $\mathcal{F}_1$ gives a trivial condition. The following functional, with $t \in \mathbb{R}$, seems also particularly interesting for further studies on our *new* or extended theory (to any Riemannian manifold)

$$\mathcal{F}(t, N) = \sum_{i=0}^{n} \int_N t^i \alpha_i = \int_N \det(1 - tA)\,\mathrm{vol}_N.$$

Remark. Isometric submanifolds of M of codimension higher than 1 may also be considered. In this case we use the orthogonal sphere bundles $S^{\perp}(N) = \{u \in TM : \ \|u\| = 1 \text{ and } u \perp TN\}$. These are Legendre n-submanifolds of SM_1 and so the study may be carried forward, on the interplay with the second fundamental form and the induced metric connection on the normal vector bundle of N.

3.5 Linear Weingarten Equations of space-forms

We are interested in the study of a generalised form of the so-called linear Weingarten equations, the term owing to the classical problem discussed e.g. in [BGG03, p. 34] for surfaces. Given scalar constants $c_0, \ldots, c_n$, we wish to find compact isometric immersed hypersurfaces $f : N \to M$ whose second fundamental form A satisfies

$$c_0 - \sigma_1(A)c_1 + \cdots + (-1)^n \sigma_n(A)c_n = 0. \tag{3.24}$$

This corresponds to $\hat{f}^*\Psi = 0$ where

$$\Psi = c_0\alpha_0 + c_1\alpha_1 + \cdots + c_n\alpha_n. \tag{3.25}$$

The study of a corresponding Poincaré-Cartan form $\Pi = \theta \wedge \Psi$ may arise from some Euler-Lagrange system if some conditions are fulfilled. This is an example of an inverse problem on exterior differential systems (we refer to [BGG03] for details). In our case, having the differential system generated by $\theta, \mathrm{d}\theta, \Psi$, what is called Monge-Ampére system, one tries to find Λ with the indicated Poincaré-Cartan form.

Now we assume M is a manifold of constant sectional curvature k.

By formulas (2.2), namely $\mathrm{d}\theta \wedge \alpha_i = 0$, and (3.3), we immediately find that $\Pi = \theta \wedge \Psi$ is closed. So locally Π is exact and we have germs of Euler-Lagrange systems.

However, we are further requiring global $SO(n)$-invariant Lagrangians, for reasons of their own relevance. It is known that the real vector subspace of invariant n-forms included in $\Omega^n_{SM_1}$

is generated by the $\alpha_0, \ldots, \alpha_n$ (cf. [BGG03, p. 33]). Notice the decomposition of the standard fibre (3.6). Thus we shall look for a solution of $\mathrm{d}\Lambda = \Pi$ of the form ($X_i \in \mathbb{R}$)

$$\Lambda = \sum X_i \alpha_i. \tag{3.26}$$

Applying the appropriate equation and an indices shift, we find the condition, $\forall 0 \leq i \leq n$,

$$iX_{i-1} - k(n-i)X_{i+1} = c_i \qquad\qquad (X_{-1} = X_{n+1} = 0). \tag{3.27}$$

We may write this as a matrix equation and so the problem remains in solving such linear system. The rank of the $n+1$-squared matrix on the left hand side acting on the vector $(X_0, \ldots, X_n)^t$ is n, for $k = 0$ or n even, and $n+1$, for n odd and $k \neq 0$. For n odd the determinant is

$$k^{\frac{n+1}{2}} 3^2 5^2 \cdots n^2.$$

Hence if we want to study a prescribed constant i_0-curvature *isolated* equation $c_0 + \sigma_{i_0}(A) = 0$, we must verify $(c_0, 0, \ldots, 0, (-1)^{i_0}, 0, \ldots, 0)$ is in the image of the linear system.

The classical Weingarten problem for surfaces in 3-dimensional space-forms shall let us see our arguments in practice. The case $n = 2$ involves volume, mean curvature H_N and the Gauss-Kronecker type curvature K_N. We may prescribe one as function of the other *two*. The Poincaré-Cartan form (3.25) corresponds to an invariant Euler-Lagrange system if and only if $c_0 = -kc_2$:

$$\Pi = \theta \wedge (-kc_2\alpha_0 + c_1\alpha_1 + c_2\alpha_2). \tag{3.28}$$

Then there is a line of solutions for Λ such as $\Lambda = c_1\alpha_0 + \frac{c_2}{2}\alpha_1$ or $\Lambda = \frac{c_2}{2}\alpha_1 - \frac{c_1}{k}\alpha_2$ (for $k \neq 0$). We see we cannot consider prescribed 1- or 2-curvature isolated equations. However, not all is lost. Suppose we want critical surfaces N to be amongst those with $H_N = H_0,\ K_N = K_0$ constants. The equation $c_0 - 2H_Nc_1 + K_Nc_2 = -2H_Nc_1 + (K_N - k)c_2 = 0$ tells us to choose $c_1 = K_0 - k$ and $c_2 = 2H_0$ for the Poincaré-Cartan form. There is no other linearly independent equation which originates from an invariant Euler-Lagrange system.

4 Observations on the Griffiths differential systems

4.1 Infinitesimal symmetries

For the following notions we recur to [BCG+91, BGG03, IL03]. We search for more properties of the Griffiths differential system.

It is easy to see there are no non-zero Cauchy characteristics of the contact structure (SM, θ), i.e. there exists no vector field $v \in \mathfrak{X}_{SM}\backslash 0$ on SM such that $v \lrcorner \mathcal{I} \subset \mathcal{I}$ where $\mathcal{I}$ is the d-closed differential ideal generated by θ. The Lie algebra $\mathfrak{g}_{\mathcal{I}}$ of infinitesimal symmetries v of $\mathcal{I}$, a set containing the Cauchy characteristics, is easy to compute formally. v is now required to satisfy $\mathcal{L}_v\mathcal{I} \subset \mathcal{I}$. On an adapted frame on SM, we let $v = \sum_{i=0}^{2n} v_ie_i$ and it is of course enough to check that $\mathcal{L}_v\theta$ renders a multiple of θ. By the Cartan formula,

$$\mathcal{L}_v\theta = \mathrm{d}(v\lrcorner\theta) + v\lrcorner\mathrm{d}\theta \in \mathcal{I} \quad\Longleftrightarrow\quad \left\{\begin{array}{l} s\,\mathrm{d}v_0(e_i) = -v_{i+n} \\ s\,\mathrm{d}v_0(e_{i+n}) = v_n \end{array}\right.,\ \forall 0 \leq i \leq n. \tag{4.1}$$

In particular $B^t\xi = \theta^\sharp$, the tautological horizontal vector field on SM, is an infinitesimal symmetry.

Recall that $\Omega^j_{SM} \subset \mathcal{I}$, $\forall j > n$, where Ω^j_{SM} is the space of j-forms. So the Euler systems $\{\theta, \mathrm{d}\theta, \alpha_i\}$ form a d-closed ideal. Now let $\mathcal{I}_{-1} = \mathcal{I}$ and let $\mathcal{I}_i$ be the ideal generated by $\mathcal{I}_{i-1} \cup \{\alpha_i\}$, for each $0 \leq i \leq n$. So that we have $\mathcal{I}_{i-1} \subset \mathcal{I}_i$. Then, if M has constant sectional curvature, besides a d-closed ideal filtration, we have a *Lie filtration* (cf. (3.3)):

$$\mathcal{L}_{\theta^\sharp}\mathcal{I}_i \subset \mathcal{I}_{i+1}. \tag{4.2}$$

We keep working with the tautological vector field in the class of constant sectional curvature metrics. Also because by adding any Lagrangian n-form Λ in the Griffiths differential system, we rarely or never seem able to decide if it admits non-vanishing infinitesimal symmetries.

So we want a $\Lambda = \sum_{i=0}^n X_i\alpha_i$ with real coefficients generating a d-closed ideal $\mathcal{J} = \{\theta, \mathrm{d}\theta, \Lambda\}$ (notice $\mathcal{J} \subset \mathcal{I}_n$), and we expect $\theta^\sharp \in \mathfrak{g}_{\mathcal{J}}$. Then we compute, also simplifying to $s = 1$:

$$\mathcal{L}_{\theta^\sharp}\Lambda = \theta^\sharp \lrcorner \mathrm{d}\Lambda = \sum_{i=0}^n X_i((i+1)\alpha_{i+1} - k(n-i+1)\alpha_{i-1}). \tag{4.3}$$

This is not a multiple of $\mathrm{d}\theta$ because there are no $e^{j(j+n)}$ in any of the α_i. So it can only be a multiple $c\Lambda$ of Λ itself for some constant c. Letting k be the sectional curvature and equating, this becomes

$$\sum_{j=0}^n (jX_{j-1} - k(n-j)X_{j+1} - cX_j)\alpha_j = 0. \tag{4.4}$$

If we put this in linear form $LX = 0$, then the determinant of the $n+1$-squared matrices, the first three being

$$\begin{bmatrix} -c & -k \\ 1 & -c \end{bmatrix} \qquad \begin{bmatrix} -c & -2k & \\ 1 & -c & -k \\ & 2 & -c \end{bmatrix} \qquad \begin{bmatrix} -c & -3k & & \\ 1 & -c & -2k & \\ & 2 & -c & -k \\ & & 3 & -c \end{bmatrix}, \tag{4.5}$$

must be

$$\det L = \begin{cases} (c^2+k)(c^2+9k)(c^2+25k)\cdots(c^2+n^2k) & \text{for } n \text{ odd} \\ -c(c^2+4k)(c^2+16k)(c^2+36k)\cdots(c^2+n^2k) & \text{for } n \text{ even} \end{cases}. \tag{4.6}$$

On the basis of such conjecture, proved up to $n = 6$, the conclusions are the following: i) there always exists a 1-dimensional subspace of solutions Λ in case n is even, with $\mathcal{L}_{\theta^\sharp}\Lambda = 0$. ii) if $k \leq 0$, then we have a 1-dimensional subspace of solutions $\mathbb{R}\Lambda_j$ satisfying

$$\mathcal{L}_{\theta^\sharp}\Lambda_j = c_j\Lambda_j \quad \forall \begin{cases} 0 \leq j \leq \frac{n-1}{2} \text{ with } c_j = \pm(2j+1)\sqrt{-k}, & \text{for } n \text{ odd} \\ 0 \leq j \leq \frac{n}{2} \text{ with } c_j = \pm(2j)\sqrt{-k}, & \text{for } n \text{ even} \end{cases}. \tag{4.7}$$

In case i) the theory tells us $\theta^\sharp \lrcorner \mathrm{d}\Lambda$ is a conservation law of the Euler-Lagrange system $\{\theta, \mathrm{d}\theta, \Lambda\}$. However this is a redundant conclusion from the definitions.

4.2 Final comments

There are a few comments on possible future directions of study.

The first would certainly be to determine how dependent is the Griffiths system on the metric and orientation of M. How can we work, for instance, on the Grassmanian bundle $G_n(TM) \to M$ of n-planes of the tangent bundle, with a given orientation o and a torsion-free linear connection on M for which o is parallel. We know that there is a canonical linear Pfaffian system on such bundle, described in [BCG$^+$91, p. 91], which is locally bundle and contact isomorphic to the metric contact manifold (SM, θ). But we still miss the Griffiths forms. If indeed we can find them, then a second important issue should turn up. Explaining how the known prolongation theory of the canonical differential system of $G_n(TM)$ (or the projectivized cotangent bundle) would be introduced on SM.

The reader may actually notice that we can define new forms with the symmetric or permutation group technique thought for the α_i. Suppose we are given the following data: a linear subgroup $G \subset GL_{n+1}(\mathbb{R})$, a linear G-structure and connection ∇ on the $n+1$-manifold M (i.e. a connection on a principal G-sub-bundle P of the frame bundle), a G-equivariant morphism B from $\mathbb{R}^{n+1}$ to a G-module W_0 and finally a tensor $\eta_0 \in \otimes^p W_0^*$ with isotropy subgroup K_0. Then we have a vector bundle $W = P \times_G W_0$ and a p-tensor η on M induced from η_0. Such p-form also carries over vertically, by pull-back, to become a tensor on the manifold W or on any fibre bundle S with fibre $G/K \subset W_0$ such that $K \subset K_0$. With the new η and the morphism B equally induced on TW, through the given connection as in (2.1), we may apply the symmetric group action to produce the forms $\eta_i = \frac{1}{p!(p-i)!}\eta \circ (B^i \wedge 1^{p-i})$. The resulting exterior differential system should be interesting to explore. The case we have been absorbed with in this article is just the more natural setting of Riemannian geometry and the tangent bundle. Just to illustrate, we take again the metric tensor g on TM and the usual map B. The metric is raised to the vertical subspace $V \subset TTM$ as g itself. Then $g \circ B \wedge 1$ is the canonical symplectic form.

We may also consider the interplay of the Griffiths forms with Finsler geometry. This goes along with the study of real functions on a sphere bundle with certain convexity properties, in order to consider, for instance, deformations of the given metric (here is a practical reason to have considered any $s > 0$ for the radius of the sphere bundle $SM \to M$). In particular, the study of functions generating homotheties i.e. positive and constant along the fibres, which induce different horizontal tangent subspaces.

The study initiated here can be developed for homogeneous Riemannian manifolds (where we know that only rank 1 symmetric spaces will preserve homogeneity along SM) and developed for the specificities of Hermitian metrics. The construction of the Griffiths system in the whole complex category is surely relevant. Finally, we aim to continue the study of submanifolds and their normal sphere bundles, referred in section 3.4, the study of the second-variation of the Lagrangians seen earlier and new ones. The most important problem however is the discovery, in some chosen context, of a set of conservation laws of the Griffiths system.

5 Remarks and proofs of main formulas

5.1 An algebraic technique

We start by recalling an algebraic tool which creates new forms from tensors. We have introduced it in the study of *gwistor* space ([Alb10, AS09, AS10]). The proofs of all assertions regarding this technique are straightforward.

Given any p-tensor η and any endomorphisms B_i of the tangent bundle we let $\eta\circ(B_1\wedge\cdots\wedge B_p)$ denote the new p-form defined by

$$\eta\circ(B_1\wedge\cdots\wedge B_p)(v_1,\ldots,v_p)=\sum_{\sigma\in S_p}\mathrm{sg}(\sigma)\,\eta(B_1v_{\sigma_1},\ldots,B_pv_{\sigma_p}). \tag{5.1}$$

This contraction obeys a simple Leibniz rule under covariant differentiation, with no minus signs attached:

$$\begin{aligned}\mathcal{D}(\eta\circ(B_1\wedge\cdots\wedge B_p))=(\mathcal{D}\eta)\circ(B_1\wedge\cdots\wedge B_p)+&\\ +\sum_{j=1}^{p}\eta\circ(B_1\wedge\cdots\wedge\mathcal{D}B_j\wedge\cdots\wedge B_p).&\end{aligned} \tag{5.2}$$

If η is a p-form, then $\eta\circ(\wedge^p1)=p!\,\eta$. For a wedge of p 1-forms we have the most important identities:

$$\begin{aligned}\eta_1\wedge\ldots\wedge\eta_p\circ(B_1\wedge\ldots\wedge B_p)\ &=\ \sum_{\sigma\in S_p}\eta_1\circ B_{\sigma_1}\wedge\ldots\wedge\eta_p\circ B_{\sigma_p}\\ &=\ \sum_{\tau\in S_p}\mathrm{sg}(\tau)\,\eta_{\tau_1}\circ B_1\wedge\ldots\wedge\eta_{\tau_p}\circ B_p.\end{aligned} \tag{5.3}$$

Notice that for a 2-form η and any endomorphism B, clearly $\eta\circ B\wedge B\,(v,w)=2\eta(Bv,Bw)$. For a 3-form and two endomorphisms B,C, then letting $\circlearrowright\!\!\!\!\!\!\bigoplus$ denote cyclic sum

$$\eta\circ(B\wedge B\wedge C)(v,w,z)=2\underset{v,w,z}{\circlearrowright\!\!\!\!\!\!\bigoplus}\eta(Bv,Bw,Cz). \tag{5.4}$$

5.2 Proofs for section 2.1

We are now ready for the proofs of the auxiliary and main formulas in section 2.

To encourage the reading which follows, we start by giving an explanation of the well-known formulas (2.2). In a way these are the defining equations of a connection in relation with the horizontal subspace H. We may always assume $w=(\mathrm{d}v_1)_x(v_2)$ where $x\in M$ and $v_1,v_2\in\mathfrak{X}_M$ are vector fields. We just need the map v_1 into TM to be defined on a neighbourhood of x. As it is not so difficult to see ([Alb11]), $w^v=((\mathrm{d}v_1)_x(v_2))^v=\nabla_{v_2}v_1$. Then since $\pi\circ v_1=1_M$ and $v_1^*\xi_x=\xi_{v_1(x)}=v_1(x)$, we find

$$\nabla^*_w\xi=(v_1^*\pi^*\nabla)_{v_2}v_1^*\xi=\nabla_{v_2}v_1=w^v.$$

Moreover, if one considers the acceleration of a curve in M, then one sees that it lies inside the sub-bundle $\ker(\nabla^*_\cdot\xi)$ if and only if the curve satisfies the equations of a geodesic (cf. [Alb11]).

Proof of (2.3) and Proposition 2.1. Recall the connection D on SM induced from ∇^*, given in section 2.1: $D=\nabla^\star-\frac{1}{2}\mathcal{R}^\xi$. To prove it is torsion-free, we may likewise compute the torsion of $\nabla^\star$. First, it is easy to see that its horizontal part is the same as $\pi^*T^\nabla=0$. Secondly, disregarding the symmetric component $\nabla^\star_y z-\nabla^*_y z$, for any two vector fields y,z on SM, cf. (2.6), the vertical part is

$$\begin{aligned}(T^{\nabla^\star}(y,z))^v &= \nabla^\star_y z^v-\nabla^\star_z y^v-[y,z]^v\\ &= \nabla^*_y\nabla^*_z\xi-\nabla^*_z\nabla^*_y\xi-\nabla^*_{[y,z]}\xi\\ &= \mathcal{R}^\xi(y,z).\end{aligned}$$

This proves (2.3). Regarding the 1-form θ defined in (2.7), we use the same connection to compute firstly:

$$\begin{aligned}(D_y\theta)z &= y(\theta(z))-\theta(D_yz)\\ &= y\langle\xi,Bz\rangle-\langle\xi,BD_yz\rangle\\ &= \langle\nabla^*_y\xi,Bz\rangle+\langle\xi,\nabla^*_yBz\rangle-\langle\xi,B\nabla^*_yz\rangle\\ &= \langle y^v,Bz\rangle.\end{aligned}$$

Hence, by a well-known formula,

$$\mathrm{d}\theta(y,z)\;=\;(D_y\theta)z-(D_z\theta)y\;=\;\langle y,Bz\rangle-\langle z,By\rangle$$

as we wished. ■

5.3 Proofs for section 2.2

Note θ is actually defined on TM. Undoubtedly θ corresponds with the pull-back of the Liouville 1-form on the cotangent bundle through the canonical isomorphism induced by the metric, as claimed in greater generality in [Alb11] for connections with torsion. And so $\mathrm{d}\theta$ corresponds with the pull-back of the (exact) canonical symplectic form of T^*M. Using the adapted direct orthonormal frame $\{e_0,e_1,\ldots,e_n,e_{n+1},\ldots,e_{2n}\}$, locally defined on SM, we prove the basic structure equations.

Proof of Proposition 2.2. First note (recall we use $e^{ab}=e^a\wedge e^b$)

$$\begin{aligned}(\mathrm{d}\theta)^i &= \sum_{j_1=1}^n e^{(n+j_1)j_1}\wedge\cdots\wedge\sum_{j_i=1}^n e^{(n+j_i)j_i}\\ &= \sum_{1\leq j_1<\ldots<j_i\leq n} i!\,e^{(n+j_1)j_1}\wedge\cdots\wedge e^{(n+j_i)j_i}\end{aligned}$$

In particular, we prove the claim

$$(\mathrm{d}\theta)^n\;=\;(-1)^{\frac{n(n+1)}{2}}n!\,e^{1\ldots n(n+1)\ldots(2n)}$$

sufficient to ensure we have a contact structure. Now

$$\begin{aligned}*(\mathrm{d}\theta)^i &= i!(-1)^{\frac{n(n+1)}{2}}\sum_{1\leq k_1<\ldots<k_{n-i}\leq n} e^0\wedge e^{(n+k_1)k_1}\wedge\cdots\wedge e^{(n+k_{n-i})k_{n-i}}\\ &= (-1)^{\frac{n(n+1)}{2}}\frac{i!}{(n-i)!s}\,\theta\wedge(\mathrm{d}\theta)^{n-i}\end{aligned}$$

This proves the first part of (2.15) and, in particular, (2.14) due to $** = 1_{\Lambda^*}$. Now, applying the second identity of (5.3) and $n_i = n_{n-i}$, we find

$$\begin{aligned}
\alpha_i &= n_i\sum_\sigma \mathrm{sg}(\sigma)\big(e^{(n+\sigma_1)}\circ B\wedge\cdots\wedge e^{(n+\sigma_{n-i})}\circ B\wedge e^{(n+\sigma_{n-i+1})}\wedge\cdots\wedge e^{(n+\sigma_n)}\big)\\
&= n_i\sum_\sigma \mathrm{sg}(\sigma)*\big(e^{\sigma_1}\wedge\cdots\wedge e^{\sigma_{n-i}}\wedge e^{(n+\sigma_{n-i+1})}\wedge\cdots\wedge e^{(n+\sigma_n)}\big)\\
&= n_i\sum_\sigma \mathrm{sg}(\sigma)\,(-1)^{in+n}\,e^0\wedge e^{\sigma_{n-i+1}}\wedge\cdots\wedge e^{\sigma_n}\wedge e^{(n+\sigma_1)}\wedge\cdots\wedge e^{(n+\sigma_{n-i})}\\
&= \frac{n_i}{s}\sum_\tau \mathrm{sg}(\tau)\,(-1)^{in+n+i(n-i)}\,\theta\wedge e^{\tau_1}\wedge\cdots\wedge e^{\tau_i}\wedge e^{(n+\tau_{i+1})}\wedge\cdots\wedge e^{(n+\tau_n)}\\
&= \frac{(-1)^{n-i}}{s}\,\theta\wedge\alpha_{n-i}.
\end{aligned}$$

where previously the τ equal the σ composed with an obvious index permutation. Notice the last equality follows by looking attentively at the second. Formulas $\mathrm{d}\theta\wedge\alpha_i = 0$, $\alpha_i\wedge\alpha_j = 0, \forall j\neq n-i$ are easy to deduce. ■

Now let us see the proof of an important result.

Proof of Theorem 2.1 in section 2.2. Recall, for all $0\leq i\leq n$,

$$\alpha_i = n_i\,\alpha\circ(B^{n-i}\wedge 1^i)$$

where $\alpha = \alpha_n = \frac{\xi}{\|\xi\|}\lrcorner\pi^{-1}\mathrm{vol}_M$ and 1 denotes the identity endomorphism of TSM.

With the torsion-free linear connection D on SM, with the adapted frame $\{e_0,\ldots,e_{2n}\}$ and its dual coframe, we are well equipped to compute $\mathrm{d}\alpha_i$. It is obtained through the well-known formula $\mathrm{d}\alpha_i = \sum_j e^j\wedge D_j\alpha_i$. We hence need the following computation: $\forall v, v_1,\ldots,v_n$ vector fields on SM,

$$\begin{aligned}
&D_v\alpha_i(v_1,\ldots,v_n) =\\
&= v\cdot(\alpha_i(v_1,\ldots,v_n)) - \sum_{k=1}^n\alpha_i(v_1,\ldots,\nabla^\star_v v_k - \frac{1}{2}\mathcal{R}^\xi(v,v_k),\ldots,v_n)\\
&= \nabla^*_v\alpha_i(v_1,\ldots,v_n) + \frac{1}{2}\sum_k\alpha_i(v_1,\ldots,\mathcal{R}^\xi(v,v_k),\ldots,v_n)\\
&= n_i(\nabla^*_v\alpha)\circ(B^{n-i}\wedge 1^i)(v_1,\ldots,v_n) + \frac{1}{2}\sum_k\alpha_i(v_1,\ldots,\mathcal{R}^\xi(v,v_k),\ldots,v_n).
\end{aligned}\tag{5.5}$$

We have used the fact that any α_i vanishes in the direction of ξ and that $\nabla^*B = \nabla^*1 = 0$. Now, since $\nabla\mathrm{vol}_M = 0$, we have $\nabla^*\pi^{-1}\mathrm{vol}_M = \nabla^*\pi^*\mathrm{vol}_M = 0$. For any $v_j\in\mathfrak{X}_{SM}$, we then find

$$\begin{aligned}
\nabla^*_{v_j}\alpha &= \Big(v_j\big(\frac{1}{\|\xi\|}\big)\xi + \frac{1}{\|\xi\|}\nabla^*_{v_j}\xi\Big)\lrcorner(\pi^{-1}\mathrm{vol}_M) + \frac{\xi}{\|\xi\|}\lrcorner(\nabla^*_{v_j}\pi^{-1}\mathrm{vol}_M)\\
&= \frac{1}{s}v_j^v\lrcorner(\pi^{-1}\mathrm{vol}_M).
\end{aligned}\tag{5.6}$$

We see by (5.5) and (5.6) that $\mathrm{d}\alpha_i$ has obvious *flat* and *curved* components. We compute separately, first, the flat part of $\mathrm{d}\alpha_i$. Proceeding by the mentioned formula,

$$\begin{aligned}
&\sum_{j=0}^{2n} e^j\wedge(\nabla^*_j\alpha)\circ(B^{n-i}\wedge 1^i) \ = \\
&\quad= \ \frac{1}{s}\sum_{j=n+1}^{2n} e^j\wedge(e_j\lrcorner\pi^{-1}\mathrm{vol}_M)\circ(B^{n-i}\wedge 1^i)\\
&\quad= \ \frac{1}{s^2}\sum_{j=n+1}^{2n}(-1)^{j-n}\,e^j\wedge\big((\xi^\flat\wedge e^{(n+1)\cdots\hat{j}\cdots(2n)})\circ(B^{n-i}\wedge 1^i)\big)\\
&\quad= \ \frac{1}{s^2}\sum_{j=1}^{n}(-1)^{j}\,e^{j+n}\wedge\big((\xi^\flat\wedge e^{(n+1)\cdots\widehat{j+n}\cdots(2n)})\circ(B^{n-i}\wedge 1^i)\big).
\end{aligned}$$

In the following step we let $B_1=\cdots=B_{n-i}=B$ and $B_{n-i+1}=\cdots=B_n=1$. By the first identity of formula (5.3), we have in particular

$$\alpha\circ(B^{n-i}\wedge 1^i)=\sum_{\sigma\in S_n}e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}. \tag{5.7}$$

With the same technique (5.3), the previous computation becomes:

$$\begin{aligned}
&= \ \frac{1}{s^2}\sum_{j=1}^{n}\sum_{\sigma\in S_n}(-1)^j\,e^{j+n}\wedge\xi^\flat\circ B_{\sigma_1}\wedge e^{n+1}\circ B_{\sigma_2}\wedge\cdots\\
&\qquad\qquad\cdots\wedge e^{j+n-1}\circ B_{\sigma_j}\wedge e^{j+n+1}\circ B_{\sigma_{j+1}}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}\\
&= \ \frac{1}{s^2}\sum_{j=1}^{n}\sum_{\sigma\in S_n:\ \sigma_1\leq n-i}\theta\wedge e^{n+1}\circ B_{\sigma_2}\wedge\cdots\wedge e^{j+n-1}\circ B_{\sigma_j}\wedge e^{j+n}\wedge\\
&\qquad\qquad\wedge e^{j+n+1}\circ B_{\sigma_{j+1}}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}
\end{aligned}$$

since $\theta=\xi^\flat\circ B$ and $\xi^\flat\circ 1=0$. Now letting $B_1=\cdots=B_{n-i-1}=B$ and $B_{n-i}=\cdots=B_n=1$, we may continue the computation:

$$\begin{aligned}
&= \ \frac{n-i}{s^2}\theta\wedge\sum_{j=1}^{n}\sum_{\tau\in S_n:\ \tau_j=n}e^{n+1}\circ B_{\tau_1}\wedge\cdots\wedge e^{j+n-1}\circ B_{\tau_{j-1}}\wedge e^{j+n}\wedge\\
&\qquad\qquad\wedge e^{j+n+1}\circ B_{\tau_{j+1}}\wedge\cdots\wedge e^{2n}\circ B_{\tau_n}.
\end{aligned}$$

Notice in case $i=n$ this expression vanishes because θ could never appear. Continuing, we get

$$\begin{aligned}
&= \ \frac{n-i}{s^2}\theta\wedge\sum_{j=1}^{n}\sum_{\tau\in S_n:\ \tau_j=n}e^{n+1}\circ B_{\tau_1}\wedge\cdots\wedge e^{j+n}\circ B_{\tau_j}\wedge\cdots\wedge e^{2n}\circ B_{\tau_n}\\
&= \ \frac{n-i}{s^2}\theta\wedge\alpha\circ(B^{n-i-1}\wedge 1^{i+1}).
\end{aligned}$$

Hence, assuming for a moment M is flat, we have deduced, cf. (2.16),

$$\mathrm{d}\alpha_i=\frac{n_i(n-i)}{s^2n_{i+1}}\,\theta\wedge\alpha_{i+1}=\frac{i+1}{s^2}\,\theta\wedge\alpha_{i+1}. \tag{5.8}$$

Now let us see the *curved* side of (5.5). First notice $\mathcal{R}^\xi(v,\ )$ vanishes on any vertical direction v. Then, writing as it is usual $R_{abcd}=\langle R^\nabla(e_a,e_b)e_c,e_d\rangle$, $\forall a,b,c,d\in\{0,\ldots,n\}$ and using the adapted frame $e_0,\ldots,e_{2n}$ on SM, we have

$$\mathcal{R}^\xi{}_{e_j}=\sum_{p=1}^n\langle R^\nabla(e_j,\ )se_0,e_p\rangle e_{p+n}=\sum_{q=0,\ p=1}^n sR_{jq0p}\,e^q\otimes e_{p+n}.$$

It is easy to see we just have to simplify the following expression, coming from (5.5), given for all $v_1,\ldots,v_n\in\mathfrak{X}_{SM}$,

$$\begin{aligned}\sum_{k=1}^n\alpha_i(v_1,\ldots,\mathcal{R}^\xi{}_{e_j}v_k,\ldots,v_n)\ &=\ \sum_{q=0,\ k,p=1}^n sR_{jq0p}\,\alpha_i(v_1,\ldots,e^q(v_k).e_{p+n},\ldots,v_n)\\ =\ &\sum_{q=0,\ k,p=1}^n\sum_{\sigma\in S_n}\frac{1}{(n-1)!}sR_{jq0p}(-1)^{k-1}e^q(v_{\sigma_1})\,\alpha_i(e_{p+n},v_{\sigma_2},\ldots,v_{\sigma_n})\\ =\ &\sum_{q=0,\ p=1}^n sR_{jq0p}\,e^q\wedge e_{p+n}\lrcorner\alpha_i\ (v_1,\ldots,v_n)\end{aligned}$$

Finally the tensors introduced in (2.16,2.17) are coherent with the computation of $\mathrm{d}\alpha_i$ from above. Indeed we have

$$\begin{aligned}\mathcal{R}^\xi\alpha_i\ &=\ \sum_{j=0}^n e^j\wedge\frac{1}{2}\sum_{k=1}^n\alpha_i(\ldots,\mathcal{R}^\xi(e_j,\ ),\ldots)\\ &=\ \sum_{0\leq j<q\leq n}\sum_{p=1}^n sR_{jq0p}\,e^{jq}\wedge e_{p+n}\lrcorner\alpha_i.\end{aligned}\tag{5.9}$$

■

We remark the previous formula may be partly simplified if we use (5.7):

$$\begin{aligned}e_{p+n}\lrcorner\alpha_i&=n_i\sum_{k=1}^n(-1)^{k-1}\sum_{\sigma\in S_n}e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e_{p+n}\lrcorner(e^{k+n}\circ B_{\sigma_k})\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}\\ &=n_i\sum_{k=1}^n\sum_{\sigma:\ \sigma_k>n-i}(-1)^{k-1}\,e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge\delta_{pk}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}\\ &=n_i\sum_{\sigma\in S_n:\ \sigma_p>n-i}(-1)^{p-1}\,e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge\widehat{e^{n+p}\circ B_{\sigma_p}}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}.\end{aligned}\tag{5.10}$$

Now let us see the cases $\mathcal{R}^\xi\alpha_0$ and $\mathcal{R}^\xi\alpha_1$, which thence have the particular expressions appearing in (2.18,2.19). Clearly $\mathcal{R}^\xi\alpha_0=0$, so this is done. Letting $i=1$ in the formula

above, we find

$$\begin{aligned}
\mathcal{R}^\xi\alpha_1 &= \sum_{0\leq j<q\leq n}\sum_{p=1}^{n} sR_{jq0p}\, e^{jq}\wedge e_{p+n}\lrcorner\alpha_1 \\
&= n_1\sum_{0\leq j<q\leq n}\sum_{p=1}^{n} sR_{jq0p}\, e^{jq}\wedge\sum_{k=1}^{n}\sum_{\sigma\in S_n:\ \sigma_k=n}(-1)^{k-1}\delta_{pk}\, e^1\wedge\cdots\widehat{e^k}\cdots\wedge e^n \\
&= (n-1)!^{-1}\sum_{0\leq j<q\leq n}\sum_{k=1}^{n} sR_{jq0k}\, e^{jq}\wedge(-1)^{k-1}(n-1)!\, e^1\wedge\cdots\widehat{e^k}\cdots\wedge e^n \\
&= \sum_{q=1}^{n} sR_{0q0q}\, e^0\wedge e^1\wedge\cdots\wedge e^q\wedge\cdots\wedge e^n \\
&= -\frac{1}{s}\mathrm{Ric}\,(\xi,\xi)\,\mathrm{vol}
\end{aligned}$$

as we wished, cf. (2.19). In the same token of previous ideas we have the following proofs.

5.4 Proofs for sections 3.1 and 3.2

Proof of (3.3) *in section 3.1.* For constant sectional curvature we immediately find from formula (5.9) above that

$$\mathcal{R}^\xi\alpha_i = -k\,\theta\wedge\sum_{q=1}^{n} e^q\wedge e_{q+n}\lrcorner\alpha_i.$$

Now the relevant component, assuming $B_1=\ldots=B_{n-i}=B$ and $B_{n-i+1}=\ldots=B_n=1$, is

$$\begin{aligned}
\sum_{q=1}^{n} e^q\wedge e_{q+n}\lrcorner\alpha_i &= \\
&= n_i\sum_{q=1}^{n} e^q\wedge e_{q+n}\lrcorner\sum_{\sigma\in S_n} e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n} \\
&= n_i\sum_{q=1}^{n}\sum e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^{n+q}\circ B_{\sigma_q}(e_{n+q})e^q\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n} \\
&= n_i\sum_{q}\sum_{\sigma:\ \sigma_q>n-i} e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^q\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n} \qquad \text{(cf. (5.10))} \\
&= n_i i\sum_{q}\sum_{\sigma:\ \sigma_q=n-i+1} e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^q\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n}.
\end{aligned}$$

Here we may change to $B_1=\ldots=B_{n-i}=B_{n-i+1}=B$ and $B_{n-i+2}=\ldots=B_n=1$ and then, resuming the computation,

$$\begin{aligned}
&= n_i i\sum_{q=1}^{n}\sum_{\tau\in S_n:\ \tau_q=n-i+1} e^{n+1}\circ B_{\tau_1}\wedge\cdots\wedge e^{n+q}\circ B_{\tau_q}\wedge\cdots\wedge e^{2n}\circ B_{\tau_n} \\
&= \frac{n_i i}{n_{i-1}}\,\alpha_{i-1} \\
&= (n-i+1)\alpha_{i-1}.
\end{aligned}$$

The formula $\mathrm{d}\alpha_i = \frac{i+1}{s^2}\theta\wedge\alpha_{i+1} + \mathcal{R}^\xi\alpha_i = \theta\wedge(\frac{i+1}{s^2}\alpha_{i+1} - k(n-i+1)\alpha_{i-1})$ follows. ∎

Now let us complete the proof of Theorem 3.2.

Proof of formula (3.8) *in section 3.2.* Clearly from (2.15), (2.16) and (2.17) we have

$$\mathrm{d}*\alpha_{n-2} = -\frac{1}{s}\theta\wedge\mathrm{d}\alpha_2 = -\frac{1}{s}\theta\wedge\mathcal{R}^\xi\alpha_2 = -\theta\wedge\sum_{1\leq j<q\leq n}\sum_{p=1}^{n} R_{jq0p}\,e^{jq}\wedge e_{p+n}\lrcorner\alpha_2.$$

As usual, here we let $B_1=\cdots=B_{n-2}=B$, $B_{n-1}=B_n=1$. Then we continue the computation:

$$\begin{aligned}
&= -n_2\theta\wedge\sum_{1\leq j<q\leq n}\sum_{p,a,b=1}^{n} R_{jq0p}\,e^{jq}\sum_{\sigma\in S_n:\ \sigma_a=n-1,\,\sigma_b=n-1} e_{p+n}\lrcorner(e^{n+1}\circ B_{\sigma_1}\wedge\cdots\wedge e^{2n}\circ B_{\sigma_n})\\
&= -\frac{1}{2}\theta\wedge\sum_{1\leq j<q\leq n}\sum_{p=1}^{n} R_{jq0p}\,e^{jq}\Big(\sum_{a<b} e_{p+n}\lrcorner e^1\wedge\cdots\wedge e^{n+a}\wedge\cdots\wedge e^{n+b}\wedge\cdots\wedge e^n\\
&\qquad\qquad + \sum_{b<a} e_{p+n}\lrcorner e^1\wedge\cdots\wedge e^{n+b}\wedge\cdots\wedge e^{n+a}\wedge\cdots\wedge e^n\Big)\\
&= \theta\wedge\sum_{1\leq j<q\leq n}\sum_{a<b}\Big(R_{jq0a}\,e^{jq}(-1)^{a+b}e^{n+b}\wedge e^1\wedge\cdots\wedge\widehat{e^a}\wedge\cdots\wedge\widehat{e^b}\wedge\cdots\wedge e^n\\
&\qquad\qquad -R_{jq0b}\,e^{jq}(-1)^{a+b}e^{n+a}\wedge e^1\wedge\cdots\wedge\widehat{e^a}\wedge\cdots\wedge\widehat{e^b}\wedge\cdots\wedge e^n\Big)\\
&= se^0\sum_{a<b}(-R_{ab0a}\,e^{n+b} + R_{ab0b}\,e^{n+a})\wedge e^{1\cdots n}\\
&= (\sum_{a<b}+\sum_{b<a})s\,R_{ab0a}\,e^{n+b}\wedge\mathrm{vol}.
\end{aligned}$$

This is the result we were searching. ∎

5.5 Proofs for section 3.4

There is a statement about the pull-back of the Lagrangians α_i by the lift $\hat{f}$ to SM of an isometric immersion $f: N\to M$ and a subsequent formula $\hat{f}^*\alpha_i = (-1)^i\sigma_i(A)\,\mathrm{vol}_N$ which is worth proving in detail.

Proof of Proposition 3.1. For the moment it is easier to work with $n-i$ instead of i. We have seen that

$$\alpha_{n-i} = n_{n-i}\sum_{\sigma\in S_n}\mathrm{sg}(\sigma)\,e^{\sigma_1}\wedge\cdots\wedge e^{\sigma_i}\wedge e^{(n+\sigma_{i+1})}\wedge\cdots\wedge e^{(n+\sigma_n)}.$$

Thence, by (3.17),

$$\begin{aligned}
&(-1)^{n-i}\,\hat{f}^*\alpha_{n-i}\\
&\quad= n_i\sum_{\sigma\in S_n}\sum_{j,k_j=1}^{n}\mathrm{sg}(\sigma)\,A^{k_1}_{\sigma_{i+1}}\cdots A^{k_{n-i}}_{\sigma_n}\,e^{\sigma_1}\wedge\cdots\wedge e^{\sigma_i}\wedge e^{k_1}\wedge\cdots\wedge e^{k_{n-i}}\\
&\quad= n_i\sum_{\sigma}\sum_{\tau\in S_{n-i}}\mathrm{sg}(\sigma)\,A^{\sigma_{\tau_{i+1}}}_{\sigma_{i+1}}\cdots A^{\sigma_{\tau_n}}_{\sigma_n}\,e^{\sigma_1}\wedge\cdots\wedge e^{\sigma_i}\wedge e^{\sigma_{\tau_{i+1}}}\wedge\cdots\wedge e^{\sigma_{\tau_n}}\\
&\quad= n_i\sum_{\sigma}\sum_{\tau\in S_{n-i}}\mathrm{sg}(\tau)\,A^{\sigma_{\tau_{i+1}}}_{\sigma_{i+1}}\cdots A^{\sigma_{\tau_n}}_{\sigma_n}\,e^{1}\wedge\cdots\wedge e^{n}
\end{aligned}$$

where in the permutations τ operate in the set $\{i+1,\ldots,n\}$. Letting $e_1,\ldots,e_n$ be a direct orthonormal basis of eigenvectors of A with eigenvalues λ_j (recall A is symmetric), we see

$$(-1)^i\,\hat{f}^*\alpha_i = n_i\sum_{\sigma\in S_n}\lambda_{\sigma_1}\cdots\lambda_{\sigma_i}\,\mathrm{vol}_N = \sum_{1\leq j_1<\cdots<j_i\leq n}\lambda_{j_1}\cdots\lambda_{j_i}\,\mathrm{vol}_N = \sigma_i(A)\,\mathrm{vol}_N$$

as we wished. ∎

The formulas above prove the statement of [BGG03, p. 34] that the pull-back in question agrees essentially with the sum of all minor determinants of the second fundamental form of the submanifold N.

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